University of Florida School of Music

Creative Professional Lecture Series - March 28, 2025 – MUB 101 at 12:55 pm

Primal Bach: Interlocking sexy primes in the completion of Contrapunctus XIV, *Kunst der Fuge*

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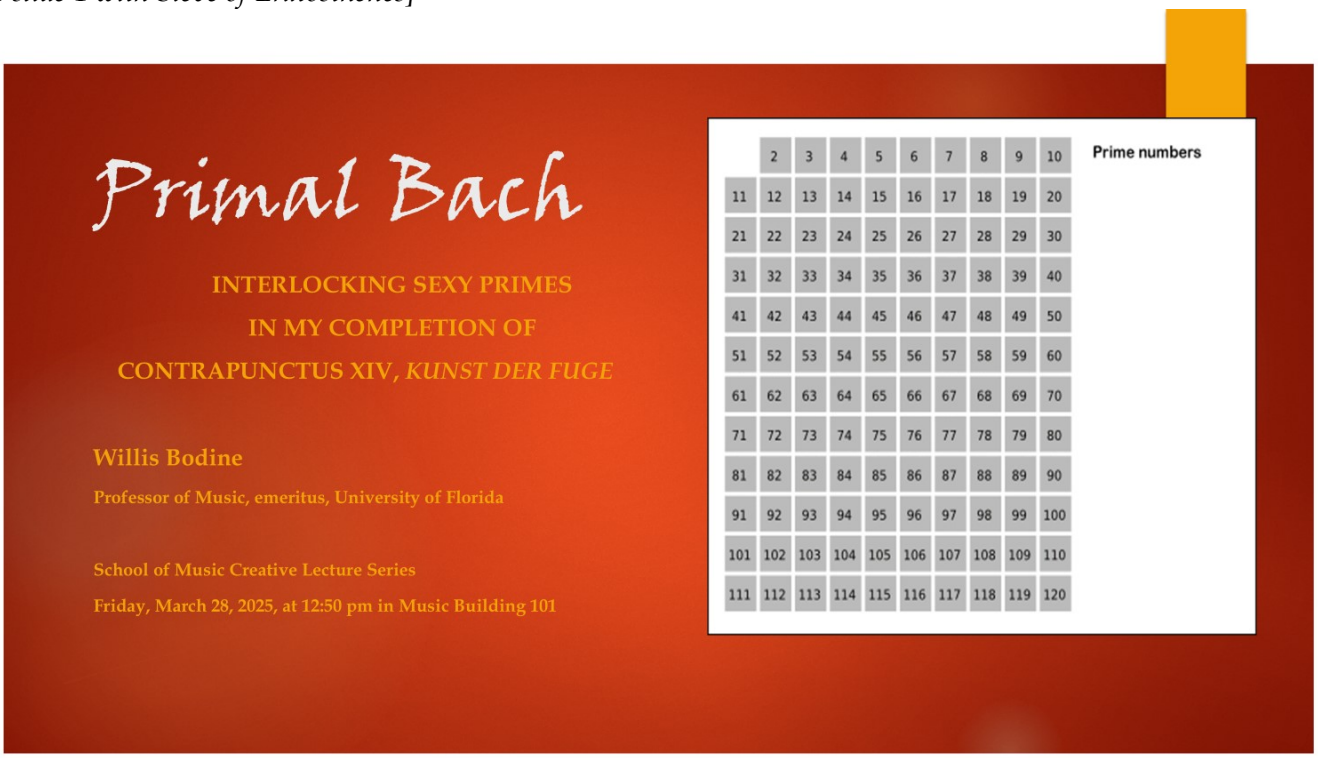
[title slide 1 with Sieve of Eratosthenes]



[At the harpsichord, Bodine plays the first bars of Contrapunctus 1, Kunst der Fuge, up to fourth subject entry]

So begins a masterwork which Johann Sebastian Bach never completed. He started it actually as a counterpoint exercise, working with his oldest son, Wilhelm Friedemann -- those first few notes *[plays]*, and so forth. They were working on a counterpoint exercise in the late 1730s.

Along about 1742 or 43, after Bach had completed the "Goldberg" variations, he put together a set of 12 fugues. *[Berlin Library, Mss. P200]* And those fugues were the beginning of the **Art of Fugue**. By the end of Bach's life, he was what, 63, 64? That's a pretty old man for that point in the 18th century. He was somewhat infirm, he had a stroke. He was growing blind, so many years of copying music in candlelight. And he was trying to sum up the career that he had had. And so, he was putting together the score of the B-minor Mass, and he was completing and publishing the **Art of Fugue**.

Like all of Bach's works, when he came back to it after six or eight years, he started to change it, he started to revise it. And so, what we have left from Bach's legacy is the beginnings of the **Art of Fugue**. He has about 80 % of the last fugue done. He has the other 18 movements done. And his sons found in his composition room pretty much of a mess, I think. The publishing was in process. The first 11 or 12 fugues were already coming

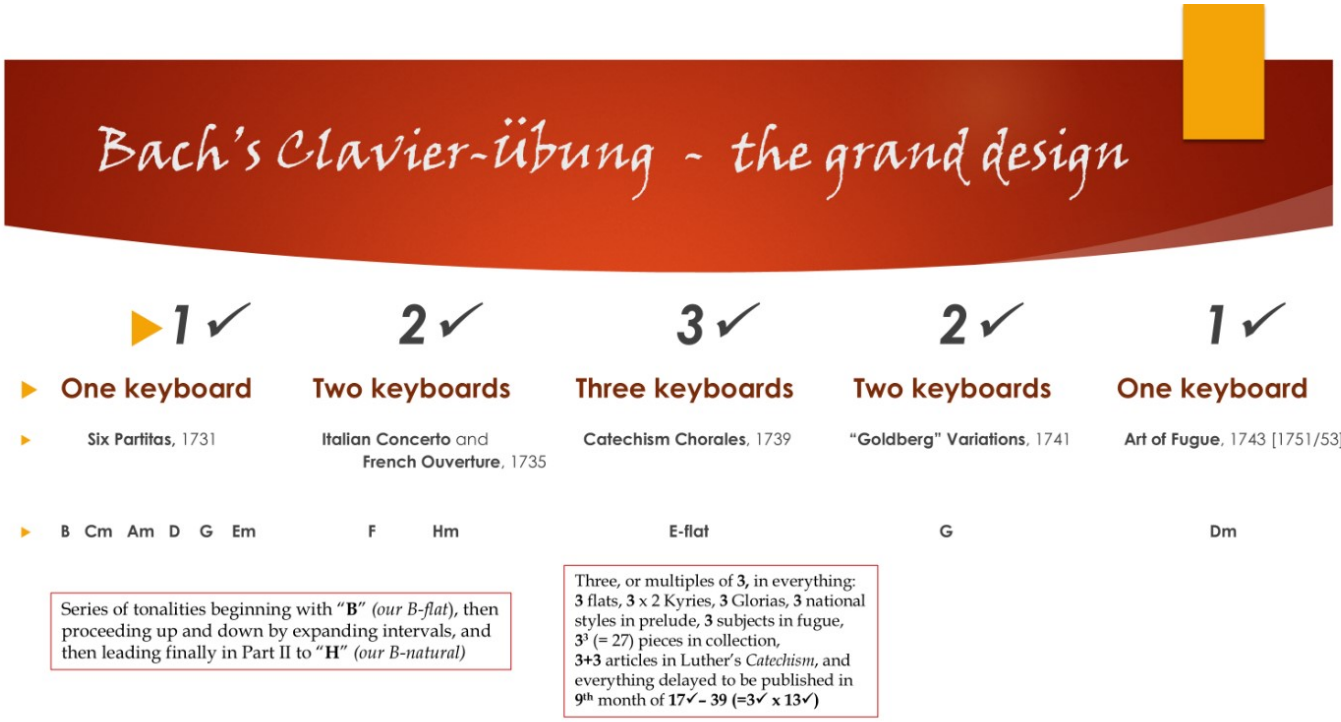
back from the printer, probably proofs scattered all over the room, miscellaneous bits and pieces, including this last little fugue. And what was left was four pages that were a good copy, and a fifth page that was not. And that's all. And so, like Schubert's *Unfinished Symphony* (which wasn't finished, that's why they call it that), like Mozart's *Requiem*, like Puccini's opera *Turandot*, it was not finished by the composer.

So, what I have done is look closely at all of the available material, the first thirteen fugues, the compositional practice that Bach used in assembling these pieces, and exactly how he proceeded in writing this 14th fugue. And so, I want to share some of that with you today.

[slide 2] For a writer, the first thing the writer does is put together the initial plan for the work. The architect figures out what he or she is going to design, where it's going to go, perhaps how it's going to be paid for. For a composer, it's not like that TV commercial with the telephone wires and the birds and a composer looks out the window and figures out the notes. That's not how it works. Composer starts with some kind of conception, or maybe even a commission if you're lucky, and then figures out where to go from there.

Our writer again has some structure in mind and perhaps starts to write some words maybe, then it is that into a book or an essay, a document of some sort, publishes it. The architect whom we left a moment ago has a program. That program tells us what the building is to do. It says we need enough space for a band, we need enough space for 19 practice rooms and so forth. And the architect continues with detailed plans. Then the architect continues, does working drawings, bidding, and construction of the process. Our composer, and this is where we are, and this is where I'll probably stop with this plan, comes the layout. The composer has to figure out how things will work. If it's a song, it has to set three stanzas of a poem. So it will have a stanza here, a little interlude, another stanza, another stanza. And so, the composer has the layout of the work.

For a fugue, the composer is following a plan that had developed over two or three centuries, in which a voice begins -- a single voice. A second voice enters in the dominant key, another voice comes in, another voice and then they proceed from there. So, Bach knew that he wanted to write a fugue. When the work continued, he would edit it, he would print it, someone would practice it, eventually it would get performed. The writer would -- let us hope -- sell the book. The architect would be pleased when we came into the building as we came in today. And, for the musician, having a performance and hearing it is the ultimate goal.



[slide 3] Bach's largest collection of organ music is the collection called "The Third Part of the C*lavier-Űbung*." The German word C*lavier-Űbung*, literally keyboard practice or exercise, was a collection of keyboard works that Bach put together. His first such publication *[of keyboard music]* from 1731 was the set of six partitas. Then the Italian Concerto, which the pianists know well, was in Part Two of 1735. The section for organ, needing three keyboards, two for the hands, one for the feet. Then the "Goldberg" Variations. And then finally, and this is my conjecture, the **Art of Fugue**. And you can see that there is a design here.

The interesting thing for me, and what has followed my researches largely since I've retired from the University, is the discovery that **composers** -- beginning in the Renaissance and continuing especially through Bach, but then into the present day, certainly in the 19th century -- **have used prime numbers as part of the layout template for their compositions**. This is actually a new discovery, nobody's come across this before. Do I need to explain prime numbers? Do I need you go out to the third grade, and you remember that prime numbers are whole numbers divisible only by themselves and by one? And that 7 is a prime number, 13 is a prime number, and 12 is not. So, in the organ section, the one for three keyboards of the C*lavier-Űbung,* Bach writes for three keyboards. It was published in the ninth month in the series, 1739. 17 and 39 are both prime numbers. In the third fugue at the end of the C*lavier-Űbung,* it has three subjects and so forth.

Bach's Clavier-übung, Part 3
(music for organ with and without pedals)

- For three (3✓) keyboards = two manuals and pedals
- Published in 9th month (=3✓x3✓) of year 17-39 (= 17✓ and 3✓x 13✓)
- Opening prelude has three (3✓) theme groups – in French, Italian and German styles
- Closing fugue ("*St Anne*") has three (3✓) subjects – the numbers of notes in each subject (6 notes & 27 notes & 15 notes) are all multiples of three (3✓)
- Prime numbers as structural template throughout – bar numbers of subject and *cantus firmus* entries, of fugal *stretti*, and of sectional divisions
- Centerpiece of Part III is the smaller setting of *Wir glauben an einen Gott* -- a movement in French style, as in WTC I and II, in the six Partitas, in the in the 4th canon of *Ein Musikalisches Opfer* and in the first version of *Kunst der Fuga*

[slide 4] The diagram is self-explanatory and perhaps leads us to realize how Bach used symbolism extensively in all of his work. The portrait from the mid -1740s. If we count the buttons on the sleeves, there are two buttons, there are three buttons on the right cuff, seven buttons on the vest, all of those are prime numbers.

Bach's use of number symbolism - I

Haußmann portrait, revised 1748

2✓ buttons on left sleeve

3✓ buttons on right cuff

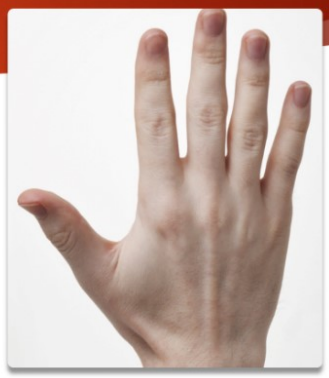
5✓ buttons on coat

7✓ buttons on vest

B - A - C - H

2✓ 1✓ 3✓ 8

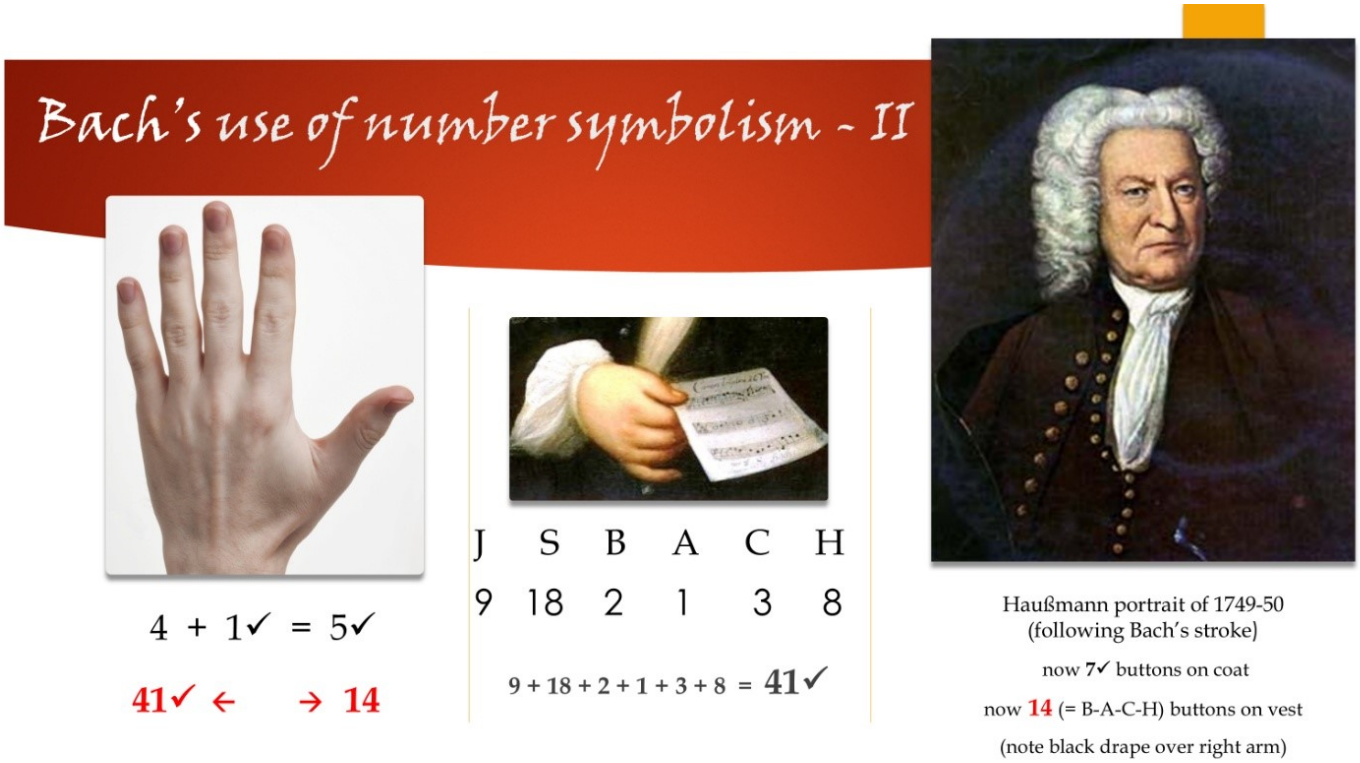
2✓ + 1✓ + 3✓ + 8 = 14



1✓ + 4 = 5✓

5✓ fingers

[slide 5] If you look at your right hand, you discover that there are four fingers. That's not prime but the thumb added to it makes five, and so five is a prime number that can be thought of in relation to the hand. Remember that -- I'll come back to it.



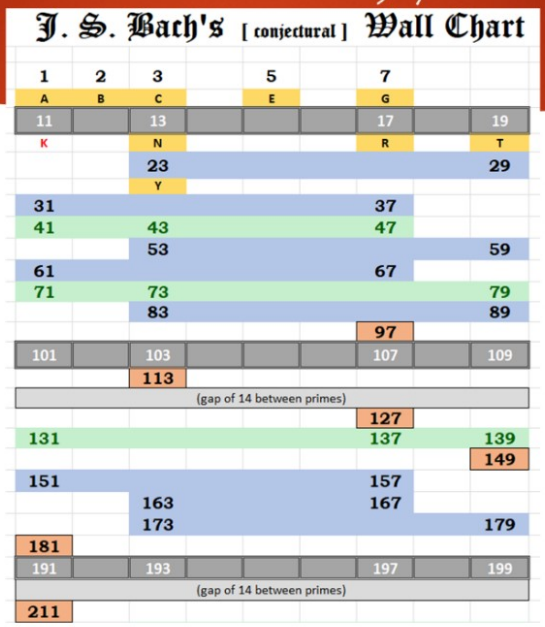
[slide 6] One of the things that composers enjoyed doing and people in general was the number-letter code. A is one, B is two, C is three and so forth, using the Roman alphabet: A, I, and J are the same, so that's 9, and U and V are the same letter. So, there are only 24 numbers involved in the number letter code. In the center of the screen, you see B, A, C, H, and that's 2 + 1 + 3 + 8 and yielding 14. Remember 14, because we'll come back to that many times. Bach certainly did and of course he was using 14 over here so it was nothing that surprised him. *[slide]* A couple of years later you may notice that he had had his stroke. There were a few more buttons on the coat, but there were still prime numbers of buttons. Now there are seven on the vest. And we go back to the number code and discover that J - S – B - A - C - H gives us 9, 18, 2, 1, 3, 8. We add them up and we get 41. 41 rather neatly is the palindrome. It's the reverse of 14 and if you look at your left hand you have four fingers and you have a thumb, and so five are true over here, and 14 and 41 become important in Bach's work. 41 is a prime number as it happens.

Number-Letter coding systems

A = 1	B = 2	C = 3	D = 4	E = 5	F = 6	G = 7	H = 8
I + J = 9	K = 10	L = 11	M = 12	N = 13	O = 14	P = 15	Q = 16
R = 17	S = 18	T = 19	U + V = 20	W = 21	X = 22	Y = 23	Z = 24

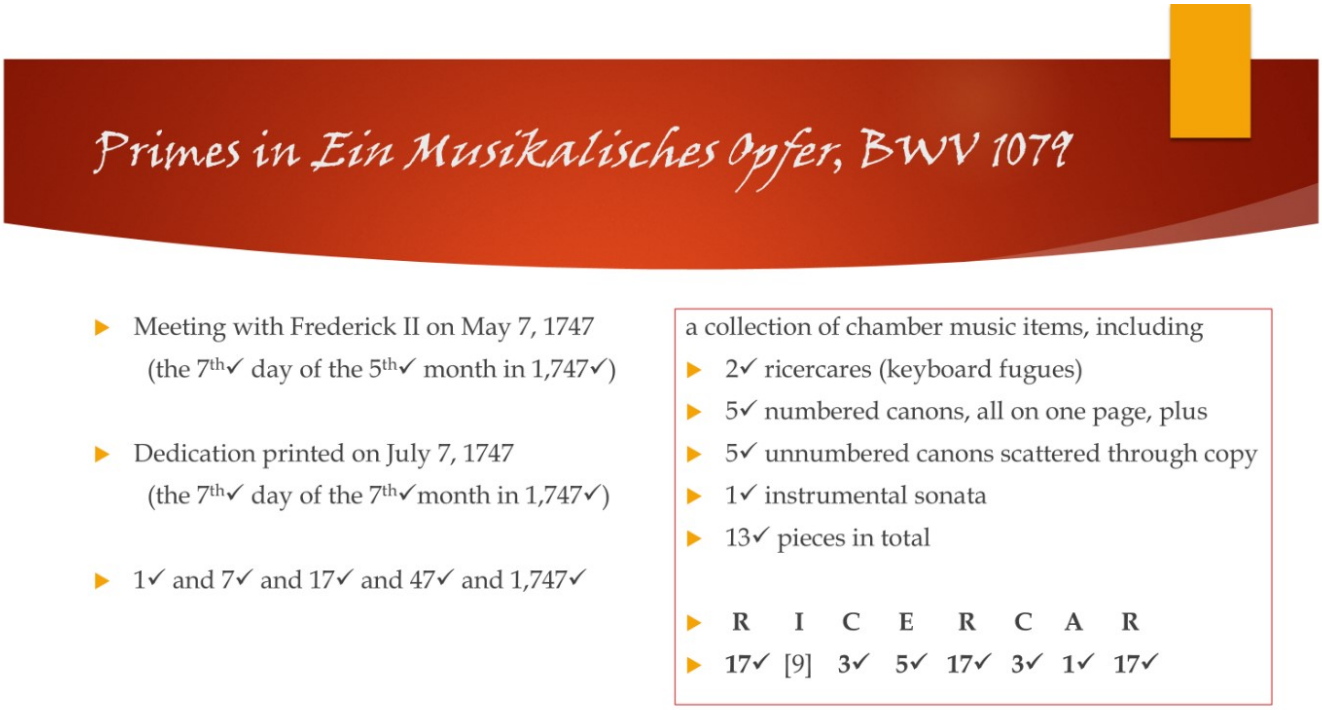
[slide 7] I don't think any of you are old enough to recognize what is on the right, so I will leave that for another day.

Bach's [conjectural] wall chart of prime numbers



[slide 8 of Bach's primes wall chart] I believe Bach had a mental wall chart and it consisted of all of the prime numbers laid out. And he also had in his mental memory, of course, the numbers and letters that applied to the numbers. So, this chart, which I won't dwell on, was a reminder when he was laying out a composition that something could happen in bar 41. Something could happen in bar 67. Something could happen in bar 29. When Bach visited Frederick the Great in 1747, he very carefully visited on May the 7th, 1747. May is the fifth month, five's a prime number, the seventh, of course, seven, 1747. I bet you've forgotten that ~~171400, sorry, I mixed it up,~~ 1,747 is a prime number. So, he was aware of what he was doing. As a thank-you note for his visit,

he wrote an incredibly complex composition called "The Musical Offering." And if we look closely through the Musical Offering, we find that there are two *ricercare* -- those are fugues -- five numbered canons, five un-numbered canons. And if you haven't figured it out by now, my little check mark is something I borrowed from the mathematicians. It means something else to them. To me, it is simply a reminder that that is a prime number when it goes by.

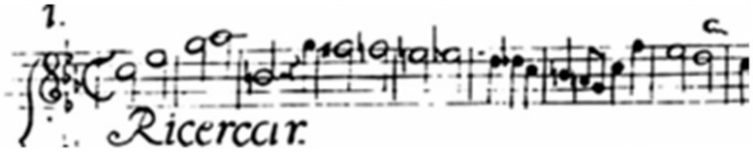


[slide 9] This is the discussion of what I have just said and so we can go forward. There is *ricercar,* that old-fashioned name for fugue. The interesting thing is that if you look closely at that, figure it out with the number code, all of the letters except for "I" (which is 9) are, in fact, prime numbers. People have wondered for years why Bach bothered to use that old-fashioned name, *ricercar*. And I suspect that this is exactly why.

The source of Bach's "Royal Theme"

- two fragments from Frederick II's *Sonata in C minor*, SpiF 190

- beginning of 3-voice Ricercar from J. S. Bach, *Ein Musikalisches Opfer, BWV* 1079



[slide 10] I believe that the story is very good that Frederick -- Frederick the Great, who was a flute player -- gave Bach the theme for improvisation. The anecdote is that he gave Bach the theme that Bach used. I believe that the story is a little different. I believe that the story is actually ~~that Bach,~~ that Frederick played a theme from one of his own flute sonatas *[plays fugal theme of 3rd movement, Sonata in C minor, SpiF 190],* and told Bach to improvise on it. I suspect Bach did but was not satisfied with it. In fact, that's what he wrote, "I'm not satisfied with this. I'm going home to work on it. I'll engrave it in copper and send it to you." And so, he did. I'll play it once more so it's in your head. This is Frederick from his flute sonata. *[plays theme at piano]* Bach changed that. I suspect the carriage ride from Potsdam to Leipzig gave him plenty of time to think about those notes. "I'll improve them," and so he did. *[plays]* Sounds more like Bach, doesn't it? Do it one more time so it's in your head. You might count the notes while they go by. *[plays]* That's five notes, *[plays]* ~~that's 11 notes.~~ I'm sorry, that's five notes more. Five notes. *[plays]* Now, I went by too fast to count, but it was 11. So, Bach constructed the theme *[of the Musical Offering]* with 5 notes plus 5 notes plus 11 notes, all of which are prime numbers.

Now we get to the subject of the day. We left Bach in his composing studio with papers littered around, not very good manuscript, and then one miserable page that dribbles off. It has a note at the bottom that his son inscribed. He said, "At this point, my father died," sort of sad. But people for decades, centuries, have tried to figure out what was going on at this point.

[slide 11] And so, the researches began. No one could figure it out. It was 130 years before people figured out that this little manuscript was actually the beginning of a quadruple fugue that was meant to conclude the entire **Art of Fugue**. A quadruple fugue will have four subjects, but this fugue that was in front of us only had three subjects. That's what they called it when they published it, a fugue with three subjects. And none of them was the **Art of Fugue** subject. So, people simply didn't know what it was. And this surprises me, it's so obvious to us. But it was 130 years before Gustav Nottebohm figured it out and published the information that, oh yes, the **Art of Fugue** main subject would in fact fit with these others, and it could be that this was the beginning of the Quadruple Fugue.

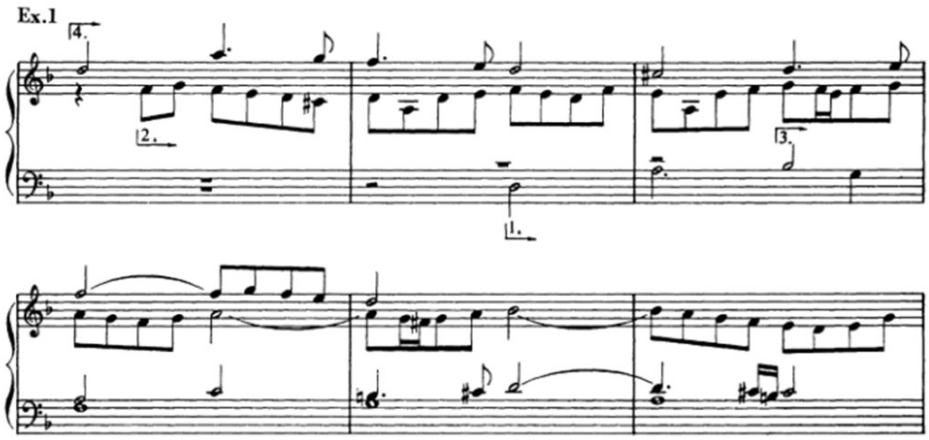
In the early 20th century, the German musician Wolfgang Graeser reawakened our interest in the **Art of Fugue** with performances and a publication. In the mid-20th century, Friedrich Smend was the writer who talked most about the number symbolism, the use of combinations of numbers and letters to symbolize things. He began looking at the 14th measure of compositions of Bach, and he would discover the musical notes B, A, C, H (B-flat, A, C, B-natural), and say, "Aha, 14th bar, B, A, C, H." There it is. Gustav Leonhardt, the notable Dutch harpsichordist, is of the opinion that the **Art of Fugue** is in fact a keyboard work. Everyone does not share that point of view. I agree with Leonhardt. I think it is a harpsichord work, mainly because it fits under the stretch of two hands, which you cannot say of other works of Bach that are meant for other combinations of instruments. The New Zealand organist Indra Hughes has talked about proportion of the sections in Contrapunctus XIV, and I'll get to that. The Danish musicologist Ruth Tatlow has talked about the way Bach managed to arrange his collections of music so that there were a thousand bars of music in this subsection and a thousand bars in this section and therefore they were parallel. This was a surprise also -- hadn't been figured out. And the Russian musicologist Antonin Milka has summarized all of that information.

- use of the **sequence of primes**, especially in (a) the number of notes in a subject, (b) the number of subject entries in a section, and (c) the bar numbers of salient events
- tablature "correction" in bar 113 of Contrapunctus XIV manuscript – actually an ***"error volens"*** intended to call attention to the prime bar start of Section Two of the quadruple fugue
- thematic content of **episodes**, and structuring of successive episodes
- **symbolism** of having exactly 55 bars in Section Three of the quadruple fugue
- **midpoint** of entire quadruple fugue at bar 143/4, with a B.A.C.H. entry there

Let's go forward. *[slide 12]* So I believe there are a number of clues that have been overlooked in looking at the **Art of Fugue**. And the first one, you can guess the most important one for me, is the sequence of primes that is used in the number of notes in every subject, the number of subject entries in a section, and the bar numbers of very salient events. I'll show you diagrams for this as coming up. The second clue is a correction in one measure of the manuscript of that. And I'll show you what it looks like in a moment. Bach actually made a mistake in copying, but it was a deliberate mistake. This is a technique that was used in the time for calling attention to something that needed further investigation. There's also a mis-spelling at one point in the publication of the Art of Fugue. There's a place where the printer has misspelled contrapunctus, and it's deliberate. It's calling attention to a detail in the piece. It's a curious thing. We don't think of it -- to make a mistake that way -- but these folks did. The episodes, those are the sections between subject entries in a fugue, have a specific thematic content, and those clues are important.

I've talked about the symbolism, and I'll go further when I get to my diagrams. In specific form, there are actually 55 bars in my completion of Contrapunctus XIV of the Art of Fugue, and you can guess why 55 is important for section 3, which is the section in which the notes B, A, C, H appear as a fugue subject. The last clue that I mention is the clue that the midpoint of the entire quadruple fugue, as I reconstructed, is at exactly bar 143-44. And if we look down in the notes in that measure, we find another B, A, C, H -- no surprise -- and also that the number 143-144 contains 14, that number that means B-A-C-H, and it is also having to do with the proportion.

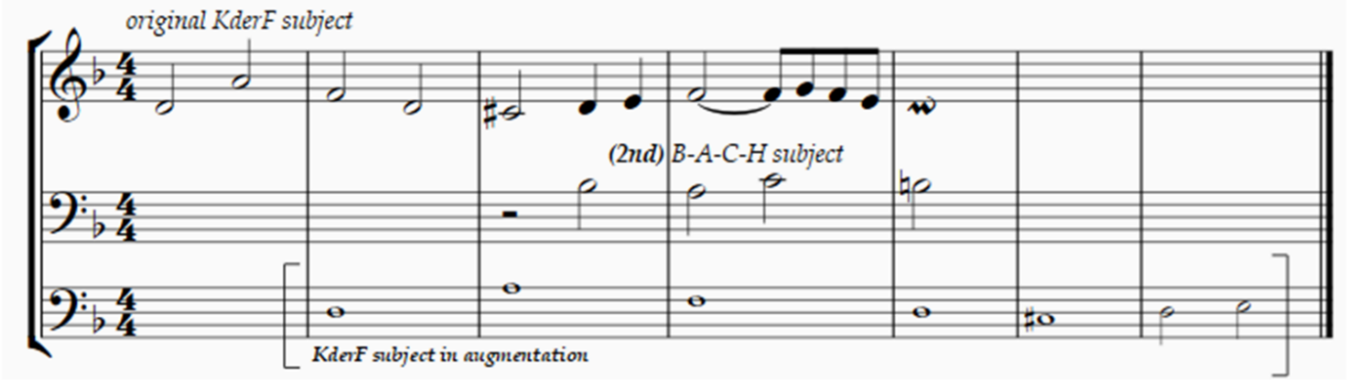
Gustav Nottebohm (1880) subjects for final fugue



So, let's look at what Nottebohm discovered. *[slide 13]* He discovered that these three subjects, this *[points]* is the first subject of the **Art of Fugue** appearing here in the bass, and here is the second subject of Contrapunctus XIV appearing in the alto, here is the third subject appearing in the tenor, and it was his discovery 130 years after Bach wrote the piece, that this fit together with the **Art of Fugue** subject -- that subject that you heard when I played the beginning of the **Art of Fugue** at the harpsichord.

KderF XIV - evolution of the quadruple fugue subjects

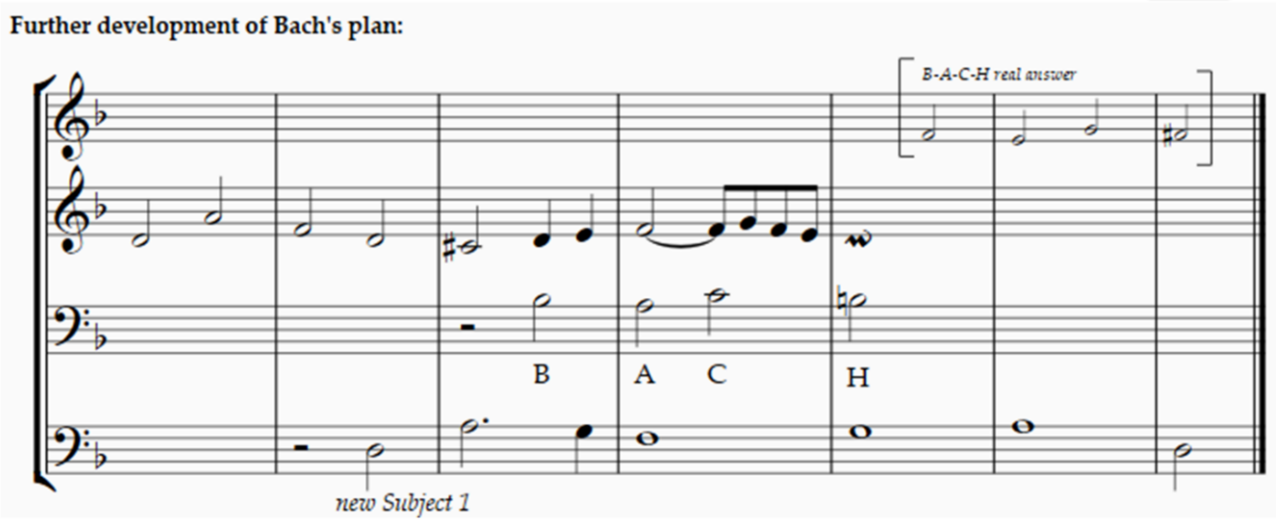
Bach's initial planning for fugue subjects:



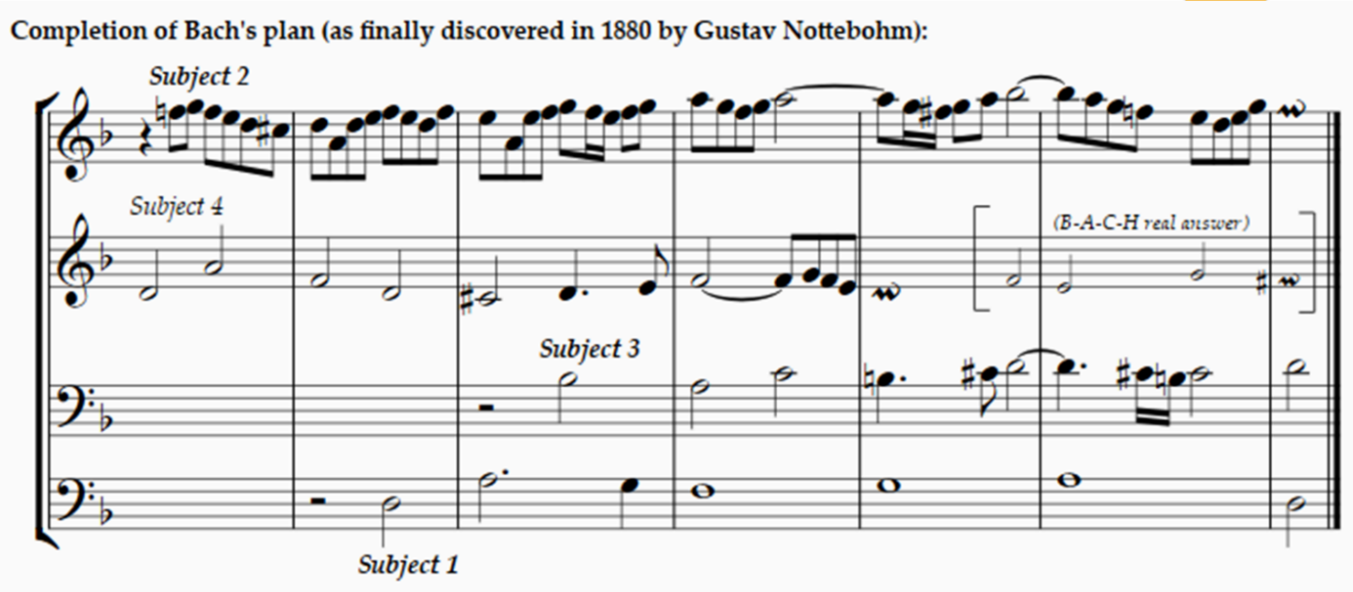
[slide 14] It's interesting to trace the evolution of those subjects. Here *[points]* is Bach starting to think about how he would lay out Contrapunctus 14. He knew he wanted to use his **Art of Fugue** subject. That was going to be a requirement, but it couldn't come till the end; no point in putting it at the beginning. We'd already heard it for 13 fugues, we might as well wait a while before we hear it in the 14th fugue. He knew also that he wanted to use his own name, and that was the subject that came together in the third section. So, putting these together makes *[an]* interesting combination. When it fits together, it works. That B -A -C -H, you're hearing it now for the first time . . . *[plays]* . . . sort of curious by itself. It's not the kind of tune that you would think of for a mid-18th century composer to use. Let's hear that combination again. *[plays]* So, Bach was on track to write his quadruple fugue. He knew what would come at the end in the fourth section. That was the big **Art of Fugue** subject. He knew what would come in the third section. That was the B - A - C - H subject.

So, we had to design a first subject and a second subject. Well, one of the techniques in fugue writing is called augmentation. You can guess: you increase the size of the notes. So, let's increase the size of that A**rt of Fugue** subject, increase the size of each note, and here it is. *[points]* And so forth. That's more than we need. Bach realized that that could become a first subject. It was suitably solid. So, were you starting to get a combination? Yes. *[plays]*

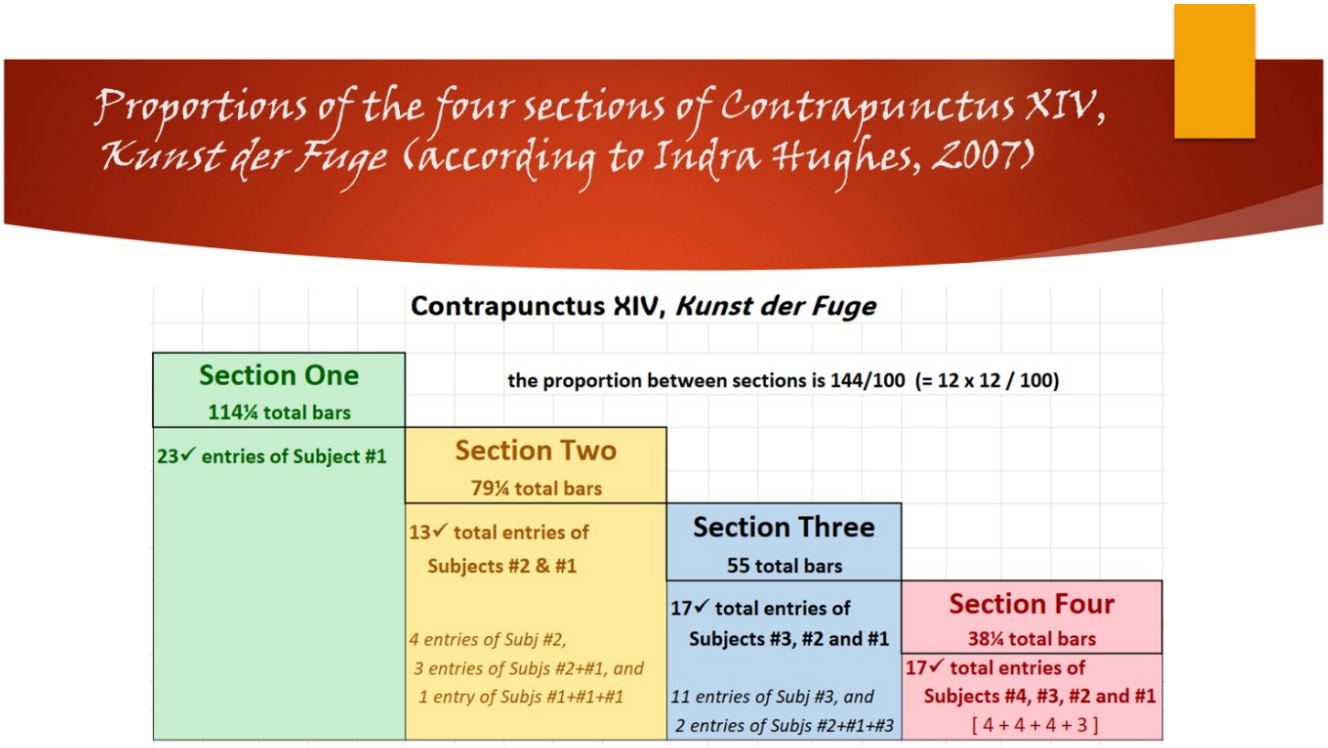
Sounds awful there at one point, that place where it went *[plays]*. That's unpleasant. However, that was the moment when a counterpoint student *[would want]* to do a suspension, which is an exciting event to have. And so the "A" *(I don't have a pointer, but I'll point to the third bar in the bottom staff)*, the "A" becomes a suspension and resolves down. And now we have, for those bottom voices . . . *[plays]* . . . and our ears are pleased to hear the suspension and its resolution.



[slide 15] So, Bach was on his way, and here is then the further development. Here is the **Art of Fugue** subject in the alto voice. Here is the new subject number one, out of an augmentation of the **Art of Fugue** subject, and here is B -A -C -H. You are right now in the mind of Bach as he was composing this piece. Sort of an exciting moment!

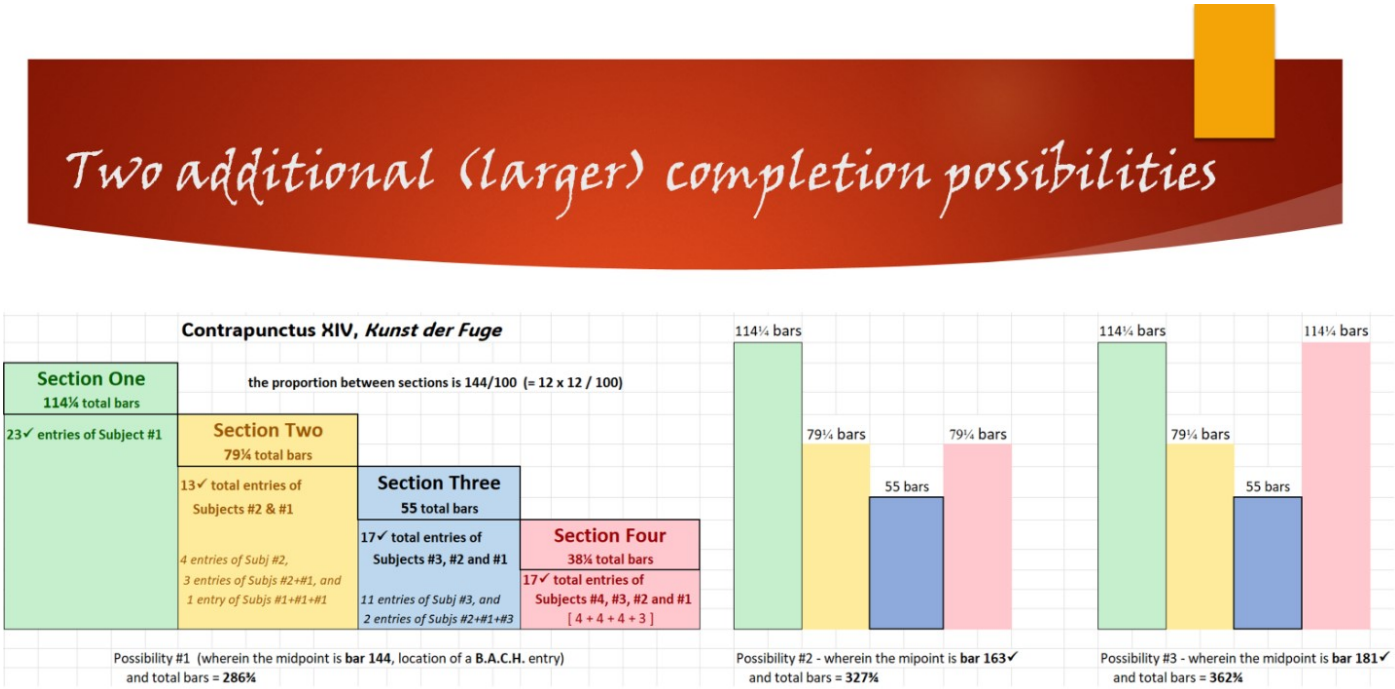


[slide 16] That's more exciting when you get the top voice, which has the rhythmic activity *[plays]* . . . and so forth. Let's see if I can put it all together *[plays]*. So, there is the working material with which Bach would set out to write his fugue.

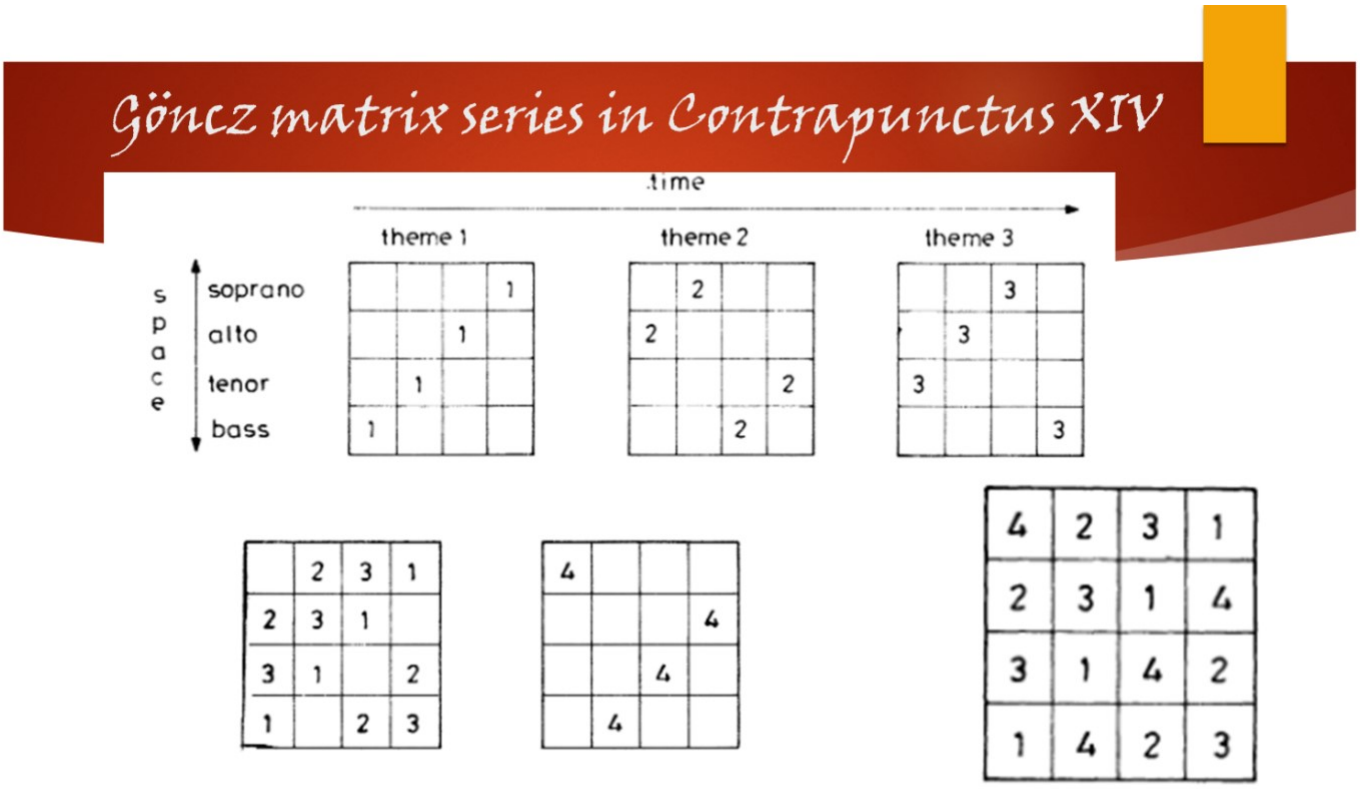


[slide 17] Next process in the layout is to figure out the proportions of Contrapuntus 14. Here is Section One as a graph. As Bach wrote it, it had 114¼ measures. Section Two had 79¼ measures. Section Three, where the BACH comes, had 55 measures -- but I have to tell the truth right now. This is the place where the manuscript breaks off. It's the place where Bach just about made clear what he was going to do, but he didn't finish it.

And then there would be the fourth section where everything came together. Now look at the shape of this diagram, and look at the next slide *[slide 18]*, because here was the judgment I made.



This *[points],* at the left, is the same thing you just saw. Here *[points]* is another way to interpret the information. Make the fourth section even bigger, and here's yet another way *[points]* to interpret the information. Make the fourth section even bigger. So, it was my judgment that what Bach did or would have done was to make the design, the layout of his fugue, symmetrical. And that's what I've done in my reconstruction.



[slide 19] There's another clue, and it has to do with the order of the entrance of the voices. And this is a diagram drawn from Göncz's book and discussion. In this diagram, here is the first exposition of the first section, where the **Art of Fugue** subject enters in the bass voice, in the tenor voice, in the alto voice and then the soprano voice. In the next section, the second theme does by Bach still enter in this pattern, in the alto, in the soprano, in the bass, in the tenor. In the third section, still by Bach, the B -A -C -H theme enters in the tenor, then in the alto, then in the soprano, then in the bass. That yields then this pattern, leaving blank spots for the fourth subject and allowing this matrix to be completed. And so this was Göncz's discovery, that Contrapuntus 14 would in fact be a matrix fugue and follow this very careful design.

First section of Contrapunctus XIV, Kunst der Fuge
(from Mss. P200, Beilage 3 – all by J. S. Bach)

Section One →

having **23✓** entries (in 114¼ bars) of **the 7✓-note Subject #1**

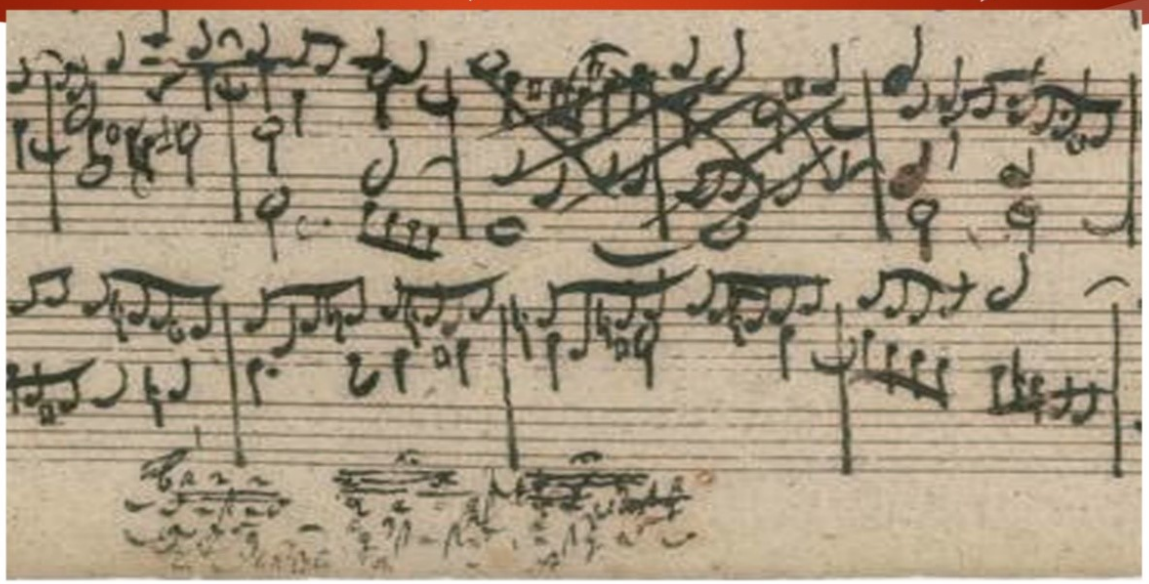
prime bars	2	7	11	17	23	29	31	37	43	47	53	61	67	71	73	79	83	89	97	107
soprano				Ans #1				Ans #1		ep		Ans #1	ep			Sinv #1			Ainv #1	
alto			Subj #1				Sinv #1	Subj #1		ep			ep	Sinv #1		Subj #1		Subj #1	Subj #1	
tenor		Ans #1			Subj #1					ep	Sinv #1		ep		Ai #1			**	Ans #1	
bass	Subj #1				Ainv #1				A #1	ep		Subj #1	ep					Subj #1		Subj #1

(blue = stretto)

** *false entry*

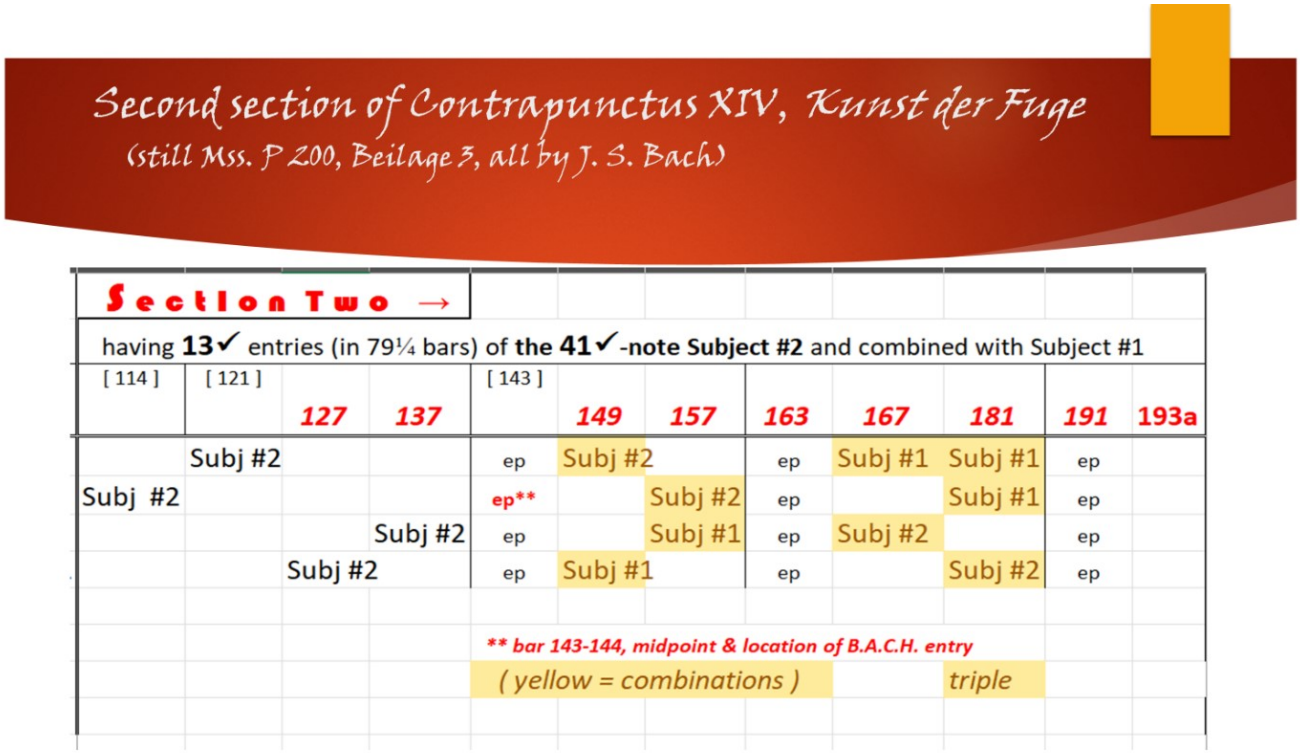
[slide 20] Now things get very interesting. Here is a breakdown of Section One -- the part that Bach wrote. The red line across here are the prime numbers of the measures in which these events happen. Subject One enters in bar 2. Subject One, the answer, enters in bar 7. You can see exactly how the diagram continues. There's a blue still show up? Yes, the blue shows well. Subjects enter in stretto. The two subjects are almost *[at]* the same time but not quite, and so for a total of 23 times we hear that first subject in Section One.

Correction in P200, Beilage 3, the 3-voice manuscript fugue which then becomes Contrapunctus XIV, Kunst der Fuge)

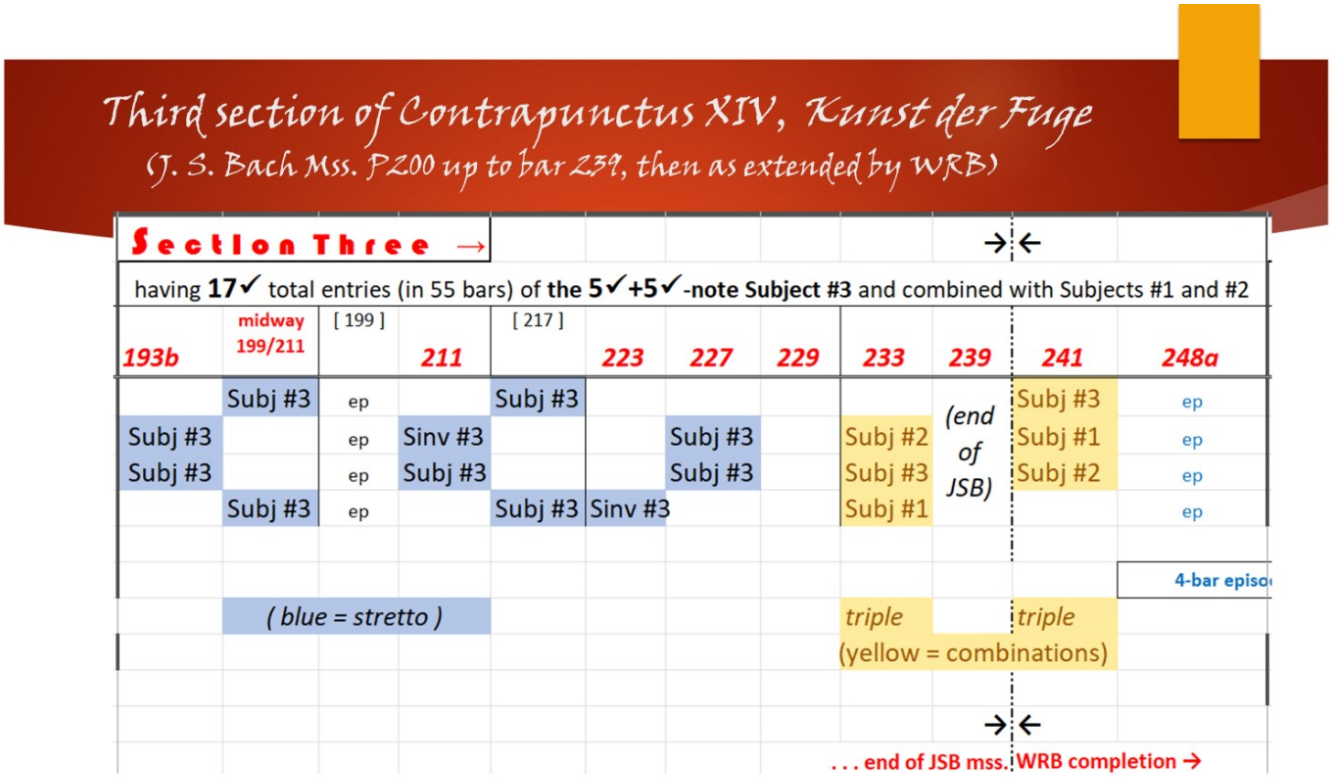


[slide 21] When Bach was copying this manuscript, he made what was an apparent error, and this is the place *[points]*. It looks as if he was simply careless, but we know better. We know how careful he was, and especially

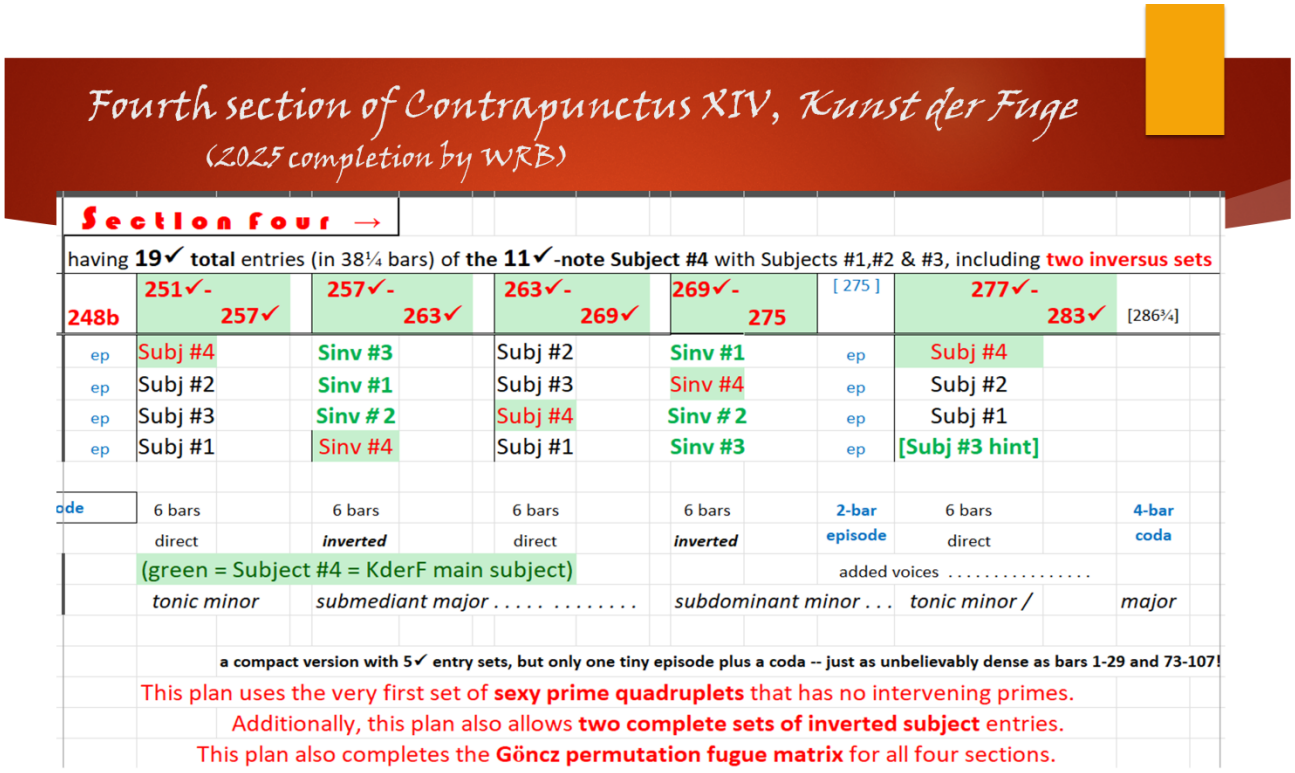
at this point with this important music. Bach, as a matter of fact, would never have written exactly this progression. It violates a primary rule, repeating a bass note across the bar line. So, clearly, there's something wrong. These two measures are marked out, and the correction appears at the bottom of the page. You can't read it, and I can't either, but it is in German organ tablature. It's a notational system that allows squibbles and letters and scrawls to represent notes and information. By deciphering that, we discover what these two measures became in three measures. And as I say, this is a clue that was meant to indicate further study.



[slide 22] The second section of Counterpoint 14, the second subject, again, *[points]* after a brief moment possibly caused by that little correction, the pattern of using prime numbers for subject entries continues and the various entries happen.



[slide 23] In Section Three, here is the place where the B -A -C -H tune comes. It comes in overlapping form or stretto immediately, and Bach gets up to the point where he begins to combine subject one, subject two, and subject three, all together. And that's where the manuscript breaks off. So, this dotted line tells you where the beginning of my reconstruction is, what I've done is do one more section combining all of those three subjects in different voices and in different places, as you see, with a short episode after it.



[slide 24] Then my completion goes on to a fourth section. Here's where things do get complicated. Again, the prime number measure numbers for subject entries are followed. The entries of all the subjects, again you see

subject four, subject inverted four, subject four, subject inverted four -- Bach is being about as complicated as he ever thought.

This section of music was described in his obituary. Apparently, the plan was being talked about, but it had not been written down or at least we've not found the manuscript yet. And then the side note, these prime numbers, 251 and 257, are what the mathematician calls sexy primes. And before anyone gets too excited, that has nothing to do with gender. It has to do with the Latin word for six, which is S – E - X, and so mathematicians call two prime numbers that are six numbers apart "sexy primes" -- which makes a good title for a lecture, I guess.

The next group of sexy primes is immediate They are linked. They are interlocked. The next group of sexy primes are linked. They are interlocked. And the last group. This is the first time in the entire list of prime numbers that there are as many as four *[consecutive]* groups of sexy primes that make a quadruplet. So, I am sure that Bach chose this carefully as the location for his subject entries in designing the subject. *Now, some of you probably have a score either in paper copy or in your phones. Is there anyone . . . It does not have either a phone or a paper copy. If you raise your hand, I see some paper copies. I don't know if these are double, but they're double -sided. Yes, they're double -sided. Anyone else? There's a couple of copies. Anyone else? This is very nostalgic for me. It used to be that we would pass handouts out in class. Maybe some people still do that. One more. Couldn't see your hand. There you go. Thank you.*

[slide 25] So, where are we in the piece? We'll go back to one. We are at the exact point that I'm pointing to. It's at the moment where subjects 1, 2, and 3, bar 233, and in your phone or in your paper copy, the boxes are the bar numbers and should keep you located. There's not much light here, so I'll do my best with these notes. This is the beginning of bar 233. *[plays harpsichord]* We're starting to hear fairly complex counterpoint. We're hearing the second subject, the active one *[plays]*, and so forth. We're hearing that solemn first subject and we're hearing the B-A–C-H. *[plays]*

So, all of that is working together very well. This is almost to the point where the manuscript trails off. I'm going to go on then, starting in bar 239. That is the place of the manuscript trails off, and you're not hearing

Bach now, but you're hearing me. *[plays harpsichord]* So, that is the transition into Section Four, and I believe it changed the slide to Section Four so you can follow that.

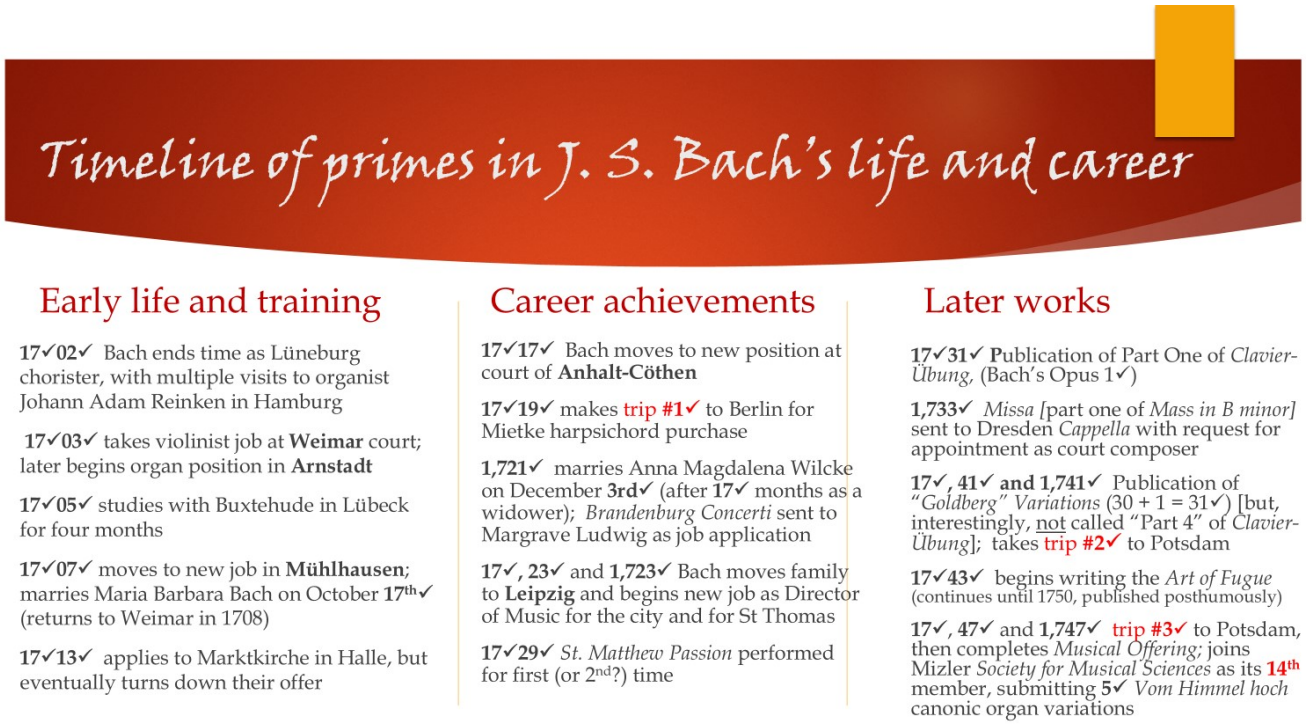
Section Four begins the **Art of Fugue** subject in the top voice. *[plays]* To hear it (this D is a little bit, a little bit tender) . . . Let's do that again. That was the first section of Section Four, which combined all four subjects. We'll keep on going. I'm starting again at bar 257. *[plays]* Well, something is wrong. The subjects don't seem to be quite the way they were. They've gotten turned upside down. And so, in this section of the piece, that bar 257, Bach has inversion of the subjects, one of the standard techniques for fugue writing. So, notice how instead of our regular **Art of Fugue** subject, we have *[plays]*. So, we'll do it at 255 *[plays]*. So, we've got one inverted section starting again at 263, and the subjects are now right side up. Right there, so . . . it'll be alright. *[Bodine fusses with instrument]* 263. *[plays]* That takes care of subjects right side up, subjects inverted, subjects right side up. So, we'll do another inverted subset and here it is, starting now at 269. *[plays]*

Now we're ready for the final section of this part of Contrapunctus XIV and this will bring everything to a conclusion, have all the subjects right side up and end. *[plays harpsichord]* At the end, for the last three bars, I did something that Bach did in most of his fugues in the **Art of Fugue**, was added an extra voice. So, there were added voices then in that last section.

Now, if you haven't studied 18th century counterpoint, you may be totally confused by that. But if you have a question that I could answer, let me answer it now, and then I have a few concluding things. Any questions that have come up that I could answer? You're totally confused. You're cowed. You've understood everything. Perfect. Every way.

I said to Dr. Birkner that I don't wear a watch, and so he is checking my time. How are we doing?

[Birkner] A couple of minutes left. Just a couple of minutes left.



[slide 26] When we look at the timeline of primes in Bach's life and career, it's interesting to see that he ended his time as a choir boy in Lüneburg. Got acquainted with Johann Adam Reincken in Hamburg, and that's actually the person from whom he probably learned the primes technique. He went on to a violin job in 1703. He took another job in Mühlhausen in 1707. He went to study with Buxtehude briefly in 1705. At any rate, you see in the diagram that I've gone through and taken the years of major events in Bach's life and broken them down by prime numbers. And it is interesting. I cannot say for certain, and he's not around so we can't ask him. But I suspect he had this in mind. Certainly, the publications of his keyboard works were in prime years. Certainly, the specific times when he visited Frederick the Great, the times when he went to Berlin to buy a harpsichord, the times when he took a new job in Leipzig, in Mühlhausen, in Anhalt -Köthen . . . So, I suspect that the primes sequence was governing his life and his projectory of plans. We cannot know, of course, for certain.

[slide 27] Again, if we had more time, we could take several other works of Bach and disassemble them in the way that I have and see that he has followed the prime sequence -- not only in the first versions but in the later versions as well.

[slide] If you've been to Amsterdam, you know why I took this low shot of the Old Church in Amsterdam. It's because surrounding the Old Church are little low office buildings with a big window in front, and in each window there is a prostitute who is offering her services to the populace. Today the Old Church is right in the middle of Amsterdam's red-light district.

In one of the churches there is a bust of Jan Sweelinck, the organist whom I mentioned, the grand-grand teacher, great-grand teacher of Johann Sebastian Bach, and his tomb is in the floor of the church. I kept my feet off of the tomb carefully.

Prime numbers as templates -- #1

Renaissance masters

1527 – Adriano Willaert, *In convertendo Domine* (Psalm 125/126)

1564 – Orlando di Lasso, *Bonjour, mon coeur*

1566 – Alesandro Striggio, *Che farà fede al cielo*

1570 – Thomas Tallis, *Spem in alium*
40-voice motet for eight 5-voice choirs
(Mary, Queen of Scots = M-A-R-I-A =
= 12 + 1✓ + 17✓ + 1✓ + 11✓ = **40**)

English virginalists and Sweelinck

1602 – Peter Phillips, transcription of di Lasso, *Bonjour, mon coeur*

1602 - Peter Phillips, transcription of Striggio, *Che farà fede al cielo*

1612 – Orlando Gibbons, *This is the record of John*

16-19✓ and **1,619✓** –
Jan Pieterzoon Sweelinck,
"Hodie Christus natus est"
(#13✓ of *Cantiones sacrae)*

North German School

1648 – Heinrich Schütz, *Geistliche Chormusik*, Op. 11✓ (29✓ motets)

1670 – Dieterich Buxtehude,
Wacht! Euch zum Streit BuxWV 100
(lengthy cantata)

Eins bitte ich vom Herrn
(Psalm 27:4), BuxWV 24

Preludes and Fugues for organ

[slide 29] Prime numbers have been used as templates, beginning in the 16th century, and I've listed some particular works where I have found prime numbers as templates. These are in publications as in the sacred choral music of Heinrich Schütz. He made it his Opus 11 *[Geistliche Chor-Music]*, and there are actually 29 motets in that collection. Sweelinck, whom I just mentioned, in 1619 published a group of sacred choral pieces. And number 13 of those -- all of these being prime numbers -- has a work which many high school choirs sing, or used to [Jan Sweelinck, *Hodie Christus natus est]* that is actually organized around prime members. So clearly, this technique is being used in various forms by various composers.

Prime numbers as templates -- #2

Bach and his circle

1707 – Passacaglia and Fugue in C minor

1707 – Cantata 4, *Christ lag in Todesbanden*

1723 – Motet III, *Jesu, meine Freude*

1724 – *Johannis-Passion*, BWV 245

1727/29 – *Matthäus-Passion*, BWV 244

1731 – *Clavier-Übung I*, Six Partitas

1747 – *Ein Musikalisches Opfer*, BWV 1079

1743-50s – *Kunst der Fuga*, BWV 1080

Wilhelm Friedemann Bach (**1750s**)

Johann Gottlieb Goldberg (**1750s**)

Mozart and Beethoven

1779 – Leopold Mozart, *Missa Solemnis in C*

1783 – Johann Georg Albrechtsberger, *Doppelfuge für Orgel*

1779 – Wolfgang Amadeus Mozart, *Vesperae de Dominica*, K. 321

Requiem, K. 626

1806 - Michael Haydn, *Requiem in B-flat*

1819 – Ludwig van Beethoven, *Missa Solemnis*
"Credo" and "Sanctus" fugues

1819 - Piano Sonata #29✓ in B-flat, Op.106

1821 - Piano Sonata #31✓ in A-flat, Op.110

1824 - Symphony No. 9 in D minor ("Choral")

1826 - Quartet, Op. 133 (= *Grosse Fuge*)

Berlioz, Brahms and Wagner

1830 - Hector Berlioz,

Sinfonie Fantastique, H.48

1856 – Richard Wagner, *Tristan und Isolde*,
Prelude to Act One and "Liebestod"

1862 – Johannes Brahms,
Ein deutsches Requiem, Op. 45

1884 Symphony No. 4, Op 98

[slide 30] I've mentioned Bach. In January, I gave a presentation which analyzed and detailed the C-minor fugue from the Passacaglia fugue *[BWV 582b],* which is also a fugue with prime numbers as its organizing element. These are various works. You do not find a major work of Mozart that follows it closely. Mozart did not seem to follow it. I haven't listed any works of Handel because he did not seem to pay any attention to it at all.

However, we got into the 19th century. Even in such a work as the opera *Tristan und Isolde* of Richard Wagner, if we look closely at the prelude, that very impressive "Tristan" chord which happens, it happens in the measure before a prime-numbered measure throughout the overture. Somebody who may have sung the *German Requiem* of Johannes Brahms -- if we had a score, I would open it and point out how the opening movement is a textbook example of following the prime-number template throughout. The first measure has a subject entry, the second measure, the third measure, he skips the fourth measure, the fifth measure has a subject entry, he skips the sixth measure, the seventh measure, and so forth through the work.

[slide 31] So, this is something that composers have continued to follow into the 20th century. The *Ludas tonalis* of Paul Hindemith, which the pianists may know, follows the sequence of prime numbers in much of this organization. This chamber work *[Carter, Sonata for Flute, Oboe, Cello and Harpsichord]*, which our faculty Baroque Ensemble performed several times of Elliott Carter, throughout has salient events in the music which are happening in prime numbers.

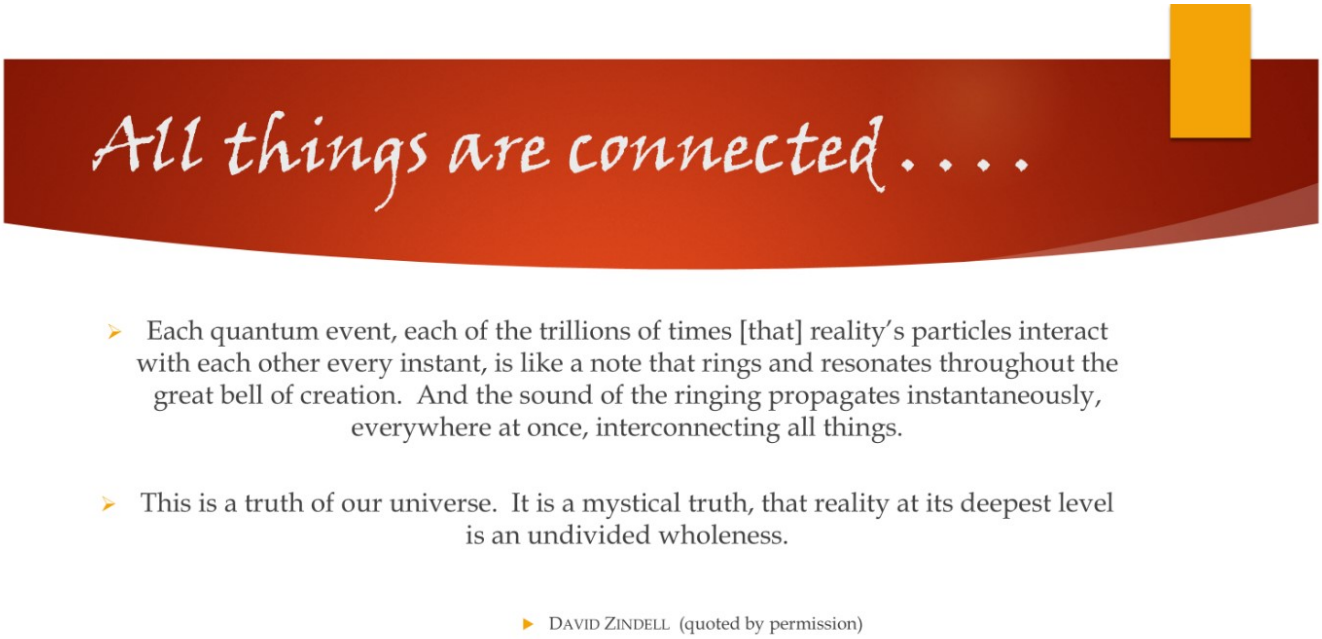
Johann Sebastian Bach's monogram and seal

[slide 32 of J. S. Bach monogram] Can you follow J? *[points]* The tip of the arrow is the J and here is the S tip of the arrow, and here is the B. There's the downstroke, and here *[points]* are the circular strokes. That's J - S - B. And then you reverse it just the same way. 14 reversed to 41, and overlay it, and you have Bach's monogram. When you go to the top of his monogram, you count one, two, three, four, five, six, the jewels across there, of which three have a corona, and then five little emblems right below it. So even in this distant place, Bach was following the prime number sequence.

Why? Why? Why? Why? Why? Why? Why?

- All these appearances of prime numbers are a complete coincidence . . . OR
  - They just happen "naturally" . . . OR
    - There is some cosmic significance . . . OR
      - The entire matter is a result of apophenia, selection bias, or confirmation bias . . . OR
        - There is some as-yet-undiscovered pattern in the sequence of primes . . . OR
          - Primes were self-evident in intellectual/musical circles of earlier times . . . OR
            - The primes technique was absorbed by creative people by observation or experience . . . OR
              - It was a deliberate design technique, passed from teacher to pupil (but was then carefully kept secret, in the ages before copyright) . . . OR
                - It is a design procedure discovered anew in each generation

[slide 33] Why was this happening? This could be a complete coincidence. It could be a very natural event, maybe something cosmic; we don't understand that. Or maybe we're just seeing it where it doesn't exist, typical apophenia of humans. Or maybe there's a pattern in prime numbers which no one has discovered yet, or maybe the primes were self-evident to intellectuals - to musical people in earlier times - and that it was absorbed by creative people, or it was a deliberate design technique passed from teacher to pupil but was then carefully kept secret in the ages before copyright. My own opinion is down toward the end of this list, and I think what is happening now is that we are discovering it in our own generation.



[slide 34] Thank you very much for your attention, and I'd be glad to talk with any of you afterwards or show any of the information. *[applause]*

(transcript of verbal lecture accompanied by PowerPoint slides)
